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Multiresolution Analyses Based on Fractal Functions

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We use a finite set of fractal interpolation functions to generate multiresolution analyses on $L^2(\mathbb{R})$ and $C_0(\mathbb{R})$. These multiresolution analyses rely on the properties of fractal functions such as self-affiniteness, existence of scaled coupled dilation equations, and the non-integral box dimension of their graph. This dimension serves as an additional parameter to better describe the small-scale structure of the set to be approximated. Concrete examples will be given to illustrate these methods.

1. INTRODUCTION

In this paper we construct multiresolution analyses that are based on fractal interpolation functions. The reason for choosing a finite set of such functions is that they are self-affine, i.e., the graph is a finite union of affine images of itself, and that they obey coupled dilation equations. The fact that the graph of a fractal interpolation function has in general a non-integral dimension d allows one to use d as an additional parameter to further specify highly complex sets. Our approach differs from the conventional one in that we use a finite set of functions to generate the multiresolution analysis. This is partly motivated by the desire to take into account the small-scale variation of functions that one wants to approximate. The structure of this paper is as follows: In Section 2 we briefly review some facts from fractal function and wavelet theory. In Section 3 we introduce the multiresolution analyses, one on $L^2(\mathbb{R})$ and one on $C_0(\mathbb{R})$. The main results are listed there. Section 4 deals primarily with the explicit construction of the set that generates the multiresolution analysis. There we apply the decomposition and reconstruction algorithms obtained from the multiresolution analyses to a concrete example.

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2. Preliminaries

In this section we review some basic definitions and results from fractal function and wavelet theory.

2.1 *Fractal Interpolation Functions*

Let $0 = x_0 < x_1 < \cdots < x_N = 1$ and $y_0, y_1, ..., y_N$ be given real numbers. For $i = 1, ..., N$ let $w_i: I \times \mathbb{R} \to I \times \mathbb{R}$ be given by

$$w_i: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a_i & 0 \\ c_i & s_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d_i \\ e_i \end{pmatrix}, \tag{2.1}$$

where $0 < |s_i| < 1$ is given, and a_i, c_i, d_i, e_i are determined by the conditions $w_i(x_0, y_0) = (x_{i-1}, y_{i-1})$ and $w_i(x_N, y_N) = (x_i, y_i)$ yielding

$$a_i = \frac{x_i - x_{i-1}}{x_N - x_0}; \tag{2.2.a}$$

$$c_i = \frac{y_i - y_{i-1}}{x_N - x_0} - \frac{s_i(y_N - y_0)}{x_N - x_0}; \tag{2.2.b}$$

$$d_i = \frac{x_N x_{i-1} - x_0 x_i}{x_N - x_0}; \tag{2.2.c}$$

$$e_i = \frac{x_N y_{i-1} - x_0 y_i}{x_N - x_0} - \frac{s_i(x_N y_0 - x_0 y_N)}{x_N - x_0}. \tag{2.2.d}$$

Note that (2.2.a) implies $0 < a_i < 1$. Let $\|\cdot\|_\theta$ be the norm on $\mathbb{R}^2$ defined by $\|(x, y)\|_\theta = |x| + \theta\,|y|$, where θ is chosen so that $0 < \theta < \min_i((1 - |a_i|)/(1 + |c_i|))$. Then it is easy to verify that w_i is a contraction in the norm $\|\cdot\|_\theta$. Let H denote the set of nonempty compact subsets of $I \times \mathbb{R}$ and h_θ the Hausdorff metric on H generated by $\|\cdot\|_\theta$. Let $W: H \to H$ be defined by

$$W(U) = \bigcup_{i=1}^{N} w_i(U) \tag{2.3}$$

for any $U \in H$. Then (see [1, 2]) it follows that W is a contraction on the complete metric space (H, h_θ). Thus the Contraction Mapping Principle implies that W has a unique fixed point $G \in H$ and that $W^{\circ n}(U) \to G$ in h_θ for any $U \in H$. We call G self-affine since it is the union of affine images of itself.

We next show that G is also the graph of a continuous function $f^* : I \to \mathbb{R}$ that interpolates the points $(x_0, y_0), ..., (x_N, y_N)$.

Let $\hat{C}(I)$ denote continuous functions on such that

$$f(x_i) = y_i, \qquad i = 0, 1, ..., N. \tag{2.4}$$

Let $u_i: I \to \mathbb{R}$ and $v_i: I \times \mathbb{R} \to \mathbb{R}$ be given by

$$\begin{aligned} u_i(x) &= a_i x + d_i \\ v_i(x, y) &= c_i x + s_i y + e_i \end{aligned} \tag{2.5}$$

for $i = 1, ..., N$ and $(x, y) \in I \times \mathbb{R}$. For $f \in \hat{C}(I)$ we define

$$\Phi(f)(x) = v_i(u_i^{-1}(x), f(u_i^{-1})(x)) \qquad \text{for} \quad x \in [x_{i-1}, x_i]. \tag{2.6}$$

It follows from (2.2) that Φ: $\hat{C}(I) \to \hat{C}(I)$. Let $s = \max |s_i| < 1$. Then (2.5) and (2.6) imply that Φ is contractive in $\| \ \|_\infty$ with contractivity s. The Contraction Mapping Principle implies that Φ has a unique fixed point $f^* \in \hat{C}(I)$. It is easy to verify that $W(\text{graph}(f^*)) = \text{graph}(f^*)$ and hence $G = \text{graph}(f^*)$; f^* is called a fractal interpolation function (FIF). Throughout this paper we work with FIF for which $\Delta x_i = (x_N - x_0)/N$, $\forall i = 1, ..., N$. We use the notation $f = (y_0, ..., y_N)_{[x_0, z_N]}$ to represent f and indicate that $f(x_i) = y_i$, on $[x_0, x_N]$, $i = 1, ..., N$.

Recall that the box dimension (sometimes also called the fractal dimension or capacity) of a bounded set $S \subseteq \mathbb{R}^n$ is defined by

$$\lim_{\varepsilon \to 0^+} \frac{\log \mathcal{N}(\varepsilon)}{\log 1/\varepsilon}, \qquad \text{provided this limit exists,}$$

where $\mathcal{N}(\varepsilon)$ denotes the minimum number of n-dimensional ε-balls needed to cover S. The box dimension d of FIF's is given by the formula

$$\sum_{i=1}^{N} |s_i| \, a_i^{d-1} = 1 \tag{2.7}$$

in the case where $\sum |s_i| > 1$ and the interpolation points are non-collinear; otherwise $d = 1$ (see [2, 3]).

2.2. *Multiresolution Analysis and Wavelets*

First we review the notion of multiresolution analysis [5, 6] and then we generalize this notion for our purposes.

For a function $\phi \in L^2(\mathbb{R})$ let $\phi_{kl}(x) = 2^{-k/2}\phi(2^{-k}x - l)$, k, $l \in \mathbb{Z}$. For $k \in \mathbb{Z}$ set

$$V_k = \text{clos}_{L^2} \text{span}\{\phi_{kl} : l \in \mathbb{Z}\}.$$

The function ϕ is said to generate a multiresolution analysis of $L^2(\mathbb{R})$ if the following conditions hold:

$$\begin{aligned}
&(\mathrm{a}) \quad \cdots \supset V_{-1} \supset V_0 \supset V_1 \supset \cdots;\\
&(\mathrm{b}) \quad \operatorname{clos}_{L^2} \textstyle\bigcup_{k\in\mathbb{Z}} V_k = L^2(\mathbb{R});\\
&(\mathrm{c}) \quad \operatorname{clos}_{L^2} \textstyle\bigcap_{k\in\mathbb{Z}} V_k = \{0\};\\
&(\mathrm{d}) \quad \forall k \in \mathbb{Z}, \qquad \{\phi_{kl} : l \in \mathbb{Z}\} \text{ is an unconditional} \qquad (2.8)\\
&\qquad \text{basis of } V_k \text{, i.e., there exists } 0 < A \leqslant B < \infty,\\
&\qquad \text{such that, } \forall (c_l)_{l\in\mathbb{Z}} \in l^2(\mathbb{Z})\\
&\qquad A\ \|(c_l)\|_{l^2(\mathbb{Z})} \leqslant \|\textstyle\sum_l c_l \phi_{kl}\|^2 \leqslant B\ \|(c_l)\|_{l^2(\mathbb{Z})}.
\end{aligned}$$

An equivalent condition to (a) under the assumption (d) is that ϕ satisfies a dilation equation of the form

$$\phi(x) = 2^{1/2} \sum_{l\in\mathbb{Z}} p_l \phi(2x-l) \tag{2.9}$$

that is, $\phi \in \operatorname{clos}_{L^2} \operatorname{span}\{\phi_{1,l} : l \in \mathbb{Z}\} = V_1$, $(p_l) \in l^2(\mathbb{Z})$.

Let $W_k = Q_k(L^2(\mathbb{R}))$ with $Q_k = P_k - P_{k-1}$, $k \in \mathbb{Z}$, where P_k denotes the orthogonal projection onto V_k. Note that all the W_k's are scaled versions of W_0, i.e.,

$$f \in W_k \Leftrightarrow f(2^k \cdot) \in W_0,$$

and that

$$\begin{aligned}
&(\mathrm{e}) \quad V_k = V_{k+1} \oplus W_{k+1}, \qquad \forall k \in \mathbb{Z};\\
&(\mathrm{f}) \quad W_k \perp W_{k'}, \qquad k \neq k'; \qquad (2.10)\\
&(\mathrm{g}) \quad L^2(\mathbb{R}) = \textstyle\bigoplus_{k\in\mathbb{Z}} W_k.
\end{aligned}$$

There exists a $\psi \in W_0$ (see [5, 6]) such that its integer-translates span W_0, i.e.,

$$W_0 = \operatorname{clos}_{L^2} \operatorname{span}\{\psi_{0,l} : l \in \mathbb{Z}\}, \tag{2.11}$$

where $\psi_{kl} = 2^{-k/2}\psi(2^{-k} \cdot - l)$, $k, l \in \mathbb{Z}$. Then

$$W_k = \operatorname{clos}_{L^2} \operatorname{span}\{\psi_{kl} : l \in \mathbb{Z}\}. \tag{2.12}$$

ψ is called a wavelet basis relative to ϕ and the W_k are called the wavelet spaces. Due to the orthogonality of the spaces W_k we may express $f_k \in V_k$ as

$$f_k = g_{k+1} + \cdots + g_{k+l} + f_{k+l}, \tag{2.13}$$

where $g_{k+j} \in W_{k+j}$, $j = 1, ..., l$, and $f_{k+l} \in V_{k+l}$. Note that $g_{k+j} = Q_{k+j} f_k$, $j = 1, ..., l$.

In case the $\{\phi_{0,l} : l \in \mathbb{Z}\}$ are orthonormal, this may be expressed recursively on $l^2(\mathbb{Z})$ as follows. Let $c(k) \in l^2(\mathbb{Z})$ and let $f_k = \sum_l c_l(k)\, \phi_{kl}$. Then

$$\begin{aligned} c_l(k+1) &= \langle f_k, \phi_{k+1,l} \rangle \\ &= \sum_{l'} p_{l'-2l} c_{l'}(k) \end{aligned} \tag{2.14}$$

Similarly, since $\psi \in V_{-1}$ there exist a $q \in l^2(\mathbb{Z})$ such that

$$\psi(x) = 2^{-1/2} \sum_l q_l \phi(2x - l). \tag{2.15}$$

Therefore, if $d(k) \in l^2(\mathbb{Z})$ and $g_k = \sum_l d_l(k)\, \psi_{kl}$,

$$d_l(k+1) = \sum_{l'} q_{l'-2l}\, d_{l'}(k). \tag{2.16}$$

3. Multiresolution Analyses Generated by Fractal Functions

In this section we use FIF's to generate a sequence of nested subspaces. Let $0 < |s| < 1$ and $N \in \mathbb{N}$, $N > 1$, be fixed throughout this section.

We define

$$\begin{aligned} \mathscr{V}_0 = \mathscr{V}_0(s, N) = \{ f : \mathbb{R} \to \mathbb{R} : \forall j \in \mathbb{Z} \\ \text{there exists a FIF } g \text{ on } [j, j+1] \text{ with } s = s_i \\ \text{and } \Delta x_i = \frac{1}{N} \text{ such that } f|_{(j,j+1)} = g|_{(j,j+1)} \}, \end{aligned} \tag{3.1}$$

and $\mathscr{V}_k$ is characterized by

$$f \in \mathscr{V}_k \Leftrightarrow f(N^{-k} \cdot) \in \mathscr{V}_0.$$

Note that $f \in \mathscr{V}_0$ is piecewise continuous with possible jump discontinuities at $x \in \mathbb{Z}$.

Theorem 1. $\cdots \supseteq \mathscr{V}_{-1} \supseteq \mathscr{V}_0 \supseteq \mathscr{V}_1 \supseteq \cdots$ *is a nested sequence of linear subspaces.*

Proof. That $\mathscr{V}_k$, $k \in \mathbb{Z}$, is a linear space follows directly from Eqs. (2.2), (2.4), (2.5), and (2.6). We remark that if $f, \hat{f} \in \mathscr{V}_k$,

$$f(x) = s f(u_i^{-1}(x)) + c_i u_i^{-1}(x) + e_i,$$

$$\hat{f}(x) = s \hat{f}(u_i^{-1}(x)) + \hat{c}_i u_i^{-1}(x) + \hat{e}_i,$$

for all $x \in u_i(I)$, and that by (2.2.b) and (2.2.d) the c_i, $\hat{c}_i$, e_i, and $\hat{e}_i$ depend linearly on $y_0, ..., y_n$. Hence the mapping $\{y_0, ..., y_n\} \to f$ is linear.

We will make use of the self-affiniteness of the graph of a FIF to show that $\mathscr{V}_k \supseteq \mathscr{V}_{k+1}$, for all $k \in \mathbb{Z}$. It suffices to prove this for $k = 0$.

Let $f \in \mathscr{V}_1$ and let without loss of generality $I = [0, N]$. Note that if $G = \text{graph } (f \mid I)$, then

$$G = \bigcup_{i'=1}^{N} w_{i'}(G) \tag{3.2}$$

implies $w_i(G) = \bigcup_{i'=1}^{N} w_i \circ w_{i'} \circ w_i^{-1}(w_i(G))$, where w_i, $i = 1, ..., N$, is as in Section 2.1. Note that $w_i \circ w_{i'} \circ w_i^{-1}$ is of the form (2.1) and that $w_i(G) = \text{graph}(f\,|_{[i-1, i]})$. Therefore, $f\,|_{[i-1, i]}$ is a FIF over $[i-1, i]$, for $i = 1, ..., N$. Hence $f \in \mathscr{V}_0$. ∎

Next we consider $\mathscr{V}_k \cap L^2(\mathbb{R})$ and $\mathscr{V}_k \cap C_0(\mathbb{R})$, $k \in \mathbb{Z}$, where $C_0(\mathbb{R})$ denotes the set of all continuous functions on $\mathbb{R}$ that vanish at infinity.

3.1. *Multiresolution Analysis of* $L^2(\mathbb{R})$

Define $V_k = \mathscr{V}_k \cap L^2(\mathbb{R})$ and let

$$e^n = \begin{cases} (0, ..., 1, ..., 0) & \text{on} \quad [0, 1] \\ 0 & \text{else,} \end{cases}$$

where e^n is an $(N+1)$-tuple describing the FIF and 1 is in the nth position, $n = 0, 1, ..., N$. It then follows that

$$V_k = \text{span}\{e^n_{kl} : n = 0, 1, ..., N;\ l \in \mathbb{Z}\} \cap L^2(\mathbb{R}) \tag{3.3}$$

There exists an orthonormal set of functions $\{\phi^0, ..., \phi^N\}$, with $\text{supp}(\phi^n) \subseteq [0, 1]$, $n = 0, 1, ..., N$, whose span equals $\text{span}\{e^n : n = 0, 1, ..., N\}$. Then $\{\phi^n_{kl} : n = 0, 1, ..., N;\ l \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. We will explicitly construct such an orthonormal set of functions in Section 4.

THEOREM 2. *The set* $\{\phi^0, ..., \phi^N\}$ *generates a multiresolution analysis of* $L^2(\mathbb{R})$ *in the following sense*:

(a) $\cdots \supset V_{-1} \supset V_0 \supset V_1 \cdots$;

(b) $\text{clos}_{L^2} \bigcup_{k \in \mathbb{Z}} V_k = L^2(\mathbb{R})$;

(c) $\bigcap_{k \in \mathbb{Z}} V_k = \{0\}$;

(d) $\forall k \in \mathbb{Z}$, *the set* $\{\phi^n_{kl} : n = 0, 1, ..., N;\ l \in \mathbb{Z}\}$ *is an unconditional basis for* V_k.

Proof. (a) Part (a) follows directly from Theorem 1.

(b) Since $(1, 1, ..., 1)_{[0,1]} = \chi_{[0,1]}$, it is clear that V_k contains $\chi_{[lN^{-k}, (l+1)N^{-k}]}$ and, therefore, $\bigcup_{k \in \mathbb{Z}} V_k$ contains all N-adic step functions in $L^2(\mathbb{R})$. Thus $\bigcup_{k \in \mathbb{Z}} V_k$ is dense in $L^2(\mathbb{R})$.

(c) Let $\mathscr{U}_i = \{f|_{[i,i+1]} : f \in V_0\}$, $i \in \mathbb{Z}$. Since $\mathscr{U}_i$ is finite-dimensional for $i \in \mathbb{Z}$, and since the norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ when restricted to $\mathscr{U}_i$ are translation-invariant they are equivalent on V_0. Thus if $f \in V_0$ then f is bounded. Furthermore, if $f \in V_k$ then f is continuous on intervals of the form $(iN^k, (i+1)N^k)$, for all $i \in \mathbb{Z}$. Thus, if $f \in \bigcap_{k \in \mathbb{Z}} V_k$ then $f \in C_0(\mathbb{R} - \{0\})$. In Theorem 3, part (c), we show that $\bigcap_{k \in \mathbb{Z}} \mathscr{V}_k \cap C_0(\mathbb{R}) = \{\text{constant functions}\}$. There we prove that if f is bounded and in $\bigcap_{k \in \mathbb{Z}} \mathscr{V}_k$, $f(x) = c_1 + c_2 H(x)$, where $c_1, c_2 \in \mathbb{R}$ and

$$H(x) = \begin{cases} 1 & x > 0 \\ -1 & x \leqslant 0. \end{cases}$$

However, since $f \in L^2(\mathbb{R})$, $c_1 = c_2 = 0$. Hence, if $f \in \bigcap_{k \in \mathbb{Z}} V_k$, then $f \equiv 0$.

(d) This follows from the fact that $\{\phi^0, ..., \phi^N\}$ is an orthonormal set of functions. ∎

As in Section 2 let W_k be the orthogonal complement of V_k in V_{k-1}. The set $\{f \in V_0 : \operatorname{supp}(f) \subseteq [0, 1]\}$ is spanned by the $N+1$ functions $\phi^0, ..., \phi^N$, the set $\{f \in W_0 : \operatorname{supp}(f) \subseteq [0, 1]\}$ by an orthonormal set $\{\psi^0, ..., \psi^{N^2-1}\}$ of $N^2 - 1$ functions. Furthermore

$$W_0 = \operatorname{clos}_{L^2} \operatorname{span}\{\psi^m_{0,l} : m = 0, 1, ..., N^2 - 1;\ l \in \mathbb{Z}\}. \tag{3.4}$$

Therefore,

$$W_k = \operatorname{clos}_{L^2} \operatorname{span}\{\psi^m_{kl} : m = 0, 1, ..., N^2 - 1;\ l \in \mathbb{Z}\}. \tag{3.5}$$

Next we set up the decomposition algorithm for this multiresolution analysis.

Let $c(0) \in l^2(\{0, 1, ..., N\} \times \mathbb{Z})$. Since $\phi^n \in V_0 \supset V_1$ and $\operatorname{supp}(\phi^n) \subseteq [0, 1]$, there exist coefficients $p^{nn'}_{l'}$, $n, n' \in \{0, 1, ..., N\}$, $l' \in \mathbb{Z}$, such that

$$\phi^n_{1,l} = \sum_{l', n'} p^{nn'}_{l'} \phi^{n'}_{0, l'+Nl}. \tag{3.6}$$

To $c(0)$ there corresponds a function $f \in V_0$ by setting

$$f = \sum_{l, n} c^n_l(0)\, \phi^n_{0,l}. \tag{3.7}$$

Let $P_1 f$ be the projection of f onto V_1, i.e.,

$$P_1 f = \sum_{l,n} c_l^n(1)\, \phi_{1,l}^n. \tag{3.8}$$

The coefficients $c_l^n(1)$, $n = 0, 1, ..., N$, $l \in \mathbb{Z}$ are obtained as follows:

$$\begin{aligned} c_l^n(1) &= \sum_{l',n'} c_{l'}^{n'}(0) \langle \phi_{0,l'}^{n'}, \phi_{1,l}^n \rangle \\ &= \sum_{l',n'} p_{l'-Nl}^{nn'} c_{l'}^{n'}(0). \end{aligned} \tag{3.9}$$

Thus we can define an operator

$$G\colon l^2(\{0, 1, ..., N\} \times \mathbb{Z}) \to l^2(\{0, 1, ..., N\} \times \mathbb{Z})$$

by

$$Gc(0) = c(1). \tag{3.10}$$

It is clear that this procedure can be iterated any number of times, and one has in general,

$$c(k+1) = Gc(k). \tag{3.11}$$

The $(c_l(k+1))_{l \in \mathbb{Z}}$ are the coefficients of f of its expansion in V_k. Similarly, if $Q_1 f$ denotes the projection of f onto W_1, we have

$$Q_1 f = \sum_{l,n} d_l^m(1)\, \psi_{1,l}^m, \qquad m = 0, 1, ..., N^2 - 1, \tag{3.12}$$

and, since $W_1 \subset V_0$, there exist coefficients $q_{l'}^{mn'}$, $m \in \{0, 1, ..., N^2 - 1\}$, $n' \in \{0, 1, ..., N\}$, $l' \in \mathbb{Z}$, such that

$$\psi_{1,l}^m = \sum_{l',n'} q_{l'}^{mn'} \phi_{0,l'+Nl}^{n'}. \tag{3.13}$$

Hence, as above, one finds

$$d_l^m(1) = \sum_{l',n'} q_{l'-Nl}^{mn'} c_{l'}^{n'}(0). \tag{3.14}$$

Thus there exists an operator $H\colon l^2(\{0, 1, ..., N\} \times \mathbb{Z}) \to l^2(\{0, 1, ..., N^2 - 1\} \times \mathbb{Z})$ such that

$$d(1) = Hc(0). \tag{3.15}$$

In general,

$$d(k+1) = Hc(k), \qquad k \in \mathbb{Z}. \tag{3.16}$$

Equations (3.11) and (3.16) constitute a "pyramid-scheme" in the sense of [5, 6, 8]. Graphically, this decomposition algorithm can be represented as in Fig. 1.

Given $c(1) \in l^2(\{0, 1, ..., N\} \times \mathbb{Z})$ and $d(1) \in l^2(\{0, 1, ..., N^2 - 1\} \times \mathbb{Z})$ we can reconstruct $c(0)$ as follows. Let

$$f = \sum_{l', n'} c_{l'}^{n'}(1)\, \phi_{1, l'}^{n'} + \sum_{l', m'} d_{l'}^{m'}(1)\, \psi_{1, l'}^{m'}.$$

Then using (3.6), (3.9), and (3.13) we obtain

$$c_l^n(0) = \langle f, \phi_{0, l}^n \rangle = \sum_{l', n'} c_{l'}^{n'}(1)\, p_{l - Nl'}^{n'n} + \sum_{l', m'} d_{l'}^{m'}(1)\, q_{l - Nl'}^{m'm}$$

or

$$c(0) = G^* c(1) + H^* d(1), \tag{3.17}$$

where G^* and H^* are the adjoints of G and H, respectively. In general, we have

$$c(k) = G^* c(k+1) + H^* d(k+1), \; k \in \mathbb{Z}. \tag{3.18}$$

3.2. *Multiresolution Analysis on $C_0(\mathbb{R})$*

Define $V_k = \mathscr{V}_k \cap C_0(\mathbb{R})$. Note that $f \in \mathscr{V}_k$ is also in V_k if it is bounded and continuous at the endpoints of the intervals $[lN^k, (l+1) N^k]$, $k, l \in \mathbb{Z}$. We can generate a basis for V_0 using the integer-translates of the functions $\phi^0 = e^0 + e_{0,-1}^N$, $\phi^j = e^j$, $j = 1, ..., N-1$, where the e^n, $n = 0, ..., N$, are defined as in Section 3.1. We then obtain

$$V_k = \operatorname{span}\{\phi_{kl}^n : n = 0, 1, ..., N-1; l \in \mathbb{Z}\} \cap C_0(\mathbb{R}) \tag{3.19}$$

(here we are dropping the normalization factor $N^{-1/2}$).

$$\begin{array}{ccccccccc} c(0) & \longrightarrow & c(1) & \longrightarrow & c(2) & \longrightarrow & \cdots & \longrightarrow & c(k) \\ & \searrow & & \searrow & & \searrow & \cdots & \searrow & \\ & & d(1) & & d(2) & & \cdots & & d(k) \end{array}$$

FIGURE 1

THEOREM 3. *The set* $\{\phi^0, ..., \phi^{N-1}\}$ *generates a multiresolution analysis of* $C_0(\mathbb{R})$ *as follows:*

(a) $\cdots V_{-1} \supset V_0 \supset V_1 \supset \cdots$;

(b) $\operatorname{clos}_{C_0(\mathbb{R})} \bigcup_{k \in \mathbb{Z}} V_k = C_0(\mathbb{R})$;

(c) $\bigcap_{k \in \mathbb{Z}} V_k = \{$*constant functions*$\}$

(d) $\forall k \in \mathbb{Z}$, *the set* $\{\phi^n_{kl} : n = 0, 1, ..., N-1, l \in \mathbb{Z}\}$ *is an unconditional basis for* V_k, *i.e., there exist* $0 < A \leqslant B < \infty$ *such that*

$$A \|c\|_{l^\infty(Z)} \leqslant \left\| \sum_{l,n} c^n_l \phi^n_{kl} \right\|_\infty \leqslant B \|c\|_{l^\infty(Z)}, \tag{3.20}$$

where $Z = \{0, 1, ..., N-1\} \times \mathbb{Z}$.

Proof. (a) Part (a) follows again from Theorem 1.

(b) By choosing the interpolation points collinear on each interval of the form $[lN^k, (l+1) N^k]$, $l, k \in \mathbb{Z}$, we generate a bounded piecewise linear function on $\mathbb{R}$. Thus V_k contains all bounded functions that are piecewise linear on $[lN^k, (l+1) N^k]$, $k, l \in \mathbb{Z}$. Since $N^k \to 0$ as $k \to -\infty$, $\bigcup_{k \in \mathbb{Z}} V_k$ is dense in $C_0(\mathbb{R})$.

(c) Clearly, any constant function is in V_k, for all $k \in \mathbb{Z}$. Let $f \in \bigcap_{k \in \mathbb{Z}} V_k$. For $k = 0, 1, 2, ...$ let Φ_k be given by (2.6) with $I = [0, N^k]$ and $y_i = f(iN^{k-1})$, $i = 0, 1, ..., N$. Let L_k be the linear interpolant through $(0, f(0))$ and $(0, f(N^k))$. Since $f \in V_{k-1} \cap V_k$ it is easy to show that $\Phi_k(L_k) |_{[0, N^{k-1}]} = L_{k-1} |_{[0, N^{k-1}]}$, and thus for any $0 \leqslant m \leqslant k$ $\Phi_m \circ \cdots \circ \Phi_k(L_k) |_{[0, N^{m-1}]} = L_m |_{[0, N^{m-1}]}$. Note that $f |_{[0, N^j]}$ is a fixed point of Φ_j for all $j = 0, 1, 2, ...$. Thus

$$\begin{aligned} &\|(f - L_m) |_{[0, N^{m-1}]}\|_\infty \\ &\quad = \|\Phi_m \circ \cdots \circ \Phi_k(f) |_{[0, N^{m-1}]} - \Phi_m \circ \cdots \circ \Phi_k(L_m) |_{[0, N^{m-1}]}\|_\infty \\ &\quad \leqslant s^{k-m+1} \|(f - L_k) |_{[0, N^k]}\|_\infty \leqslant 2s^{k-m+1} \|f\|_\infty. \end{aligned}$$

Letting $k \to \infty$ we get $f |_{[0, N^{m-1}]} = L_m |_{[0, N^{m-1}]}$, and thus $f |_{[0, \infty)} = f(0)$. Similarly, one shows $f |_{(-\infty, 0]} = f(0)$. The result now follows.

(d) Let

$$A = \min \left\{ \left\| \sum_{n=0}^{N} c^n e^n \right\|_\infty : \max_{0 \leqslant n \leqslant N} |c^n| = 1 \right\} > 0,$$

and

$$B = \max \left\{ \left\| \sum_{n=0}^{N} c^n e^n \right\|_\infty : \max_{0 \leqslant n \leqslant N} |c^n| = 1 \right\}. \quad \blacksquare$$

We define subspaces W_k so that $V_k \oplus W_k = V_{k-1}$, $k \in \mathbb{Z}$. A possible choice for W_0 is given by

$$W_0 = \operatorname{span}\{\phi^n_{-1,l} : n = 1, ..., N-1;\ l \in \mathbb{Z}\}. \tag{3.21}$$

Thus we can choose the $\phi^n_{-1,l}$, that are supported on $[0, 1]$ and their integer-translates to generate W_0. Hence we set

$$\psi^m = \phi^n_{-1,l}, \tag{3.22}$$

where $m = l(N-1) + n$, $n = 1, ..., N-1$, and $l = 0, 1, ..., N-1$. Then

$$W_k = \operatorname{span}\{\psi^m_{kl} : m = 1, ..., N(N-1);\ l \in \mathbb{Z}\}. \tag{3.23}$$

Note that for any $f \in V_0$

$$f = \sum_{l,n} f\left(l + \frac{n}{N}\right) \phi^n_{0,l} \tag{3.24}$$

and in particular,

$$\phi^n_{1,0} = \sum_{l',n'} \phi^n_{1,0}\left(l' + \frac{n'}{N}\right) \phi^{n'}_{0,l'}. \tag{3.25}$$

Define

$$p^{nn'}_{l'} = \phi^n_{1,0}\left(l' + \frac{n'}{N}\right) = \phi^n\left(\frac{l'}{N} + \frac{n'}{N^2}\right), \tag{3.26}$$

with $n = 1, ..., N-1$; $n' = 0, 1, ..., N-1$, and $l' \in \mathbb{Z}$.

Let us now look at the decomposition and reconstruction algorithm. Suppose $c(0) \in l^\infty(Z)$ and let $f = \sum_{l,n} c^n_l(0)\, \phi^n_{0,l}$. Then as above there exists the decomposition

$$\sum_{l,n} c^n_l(0)\, \phi^n_{0,l} = \sum_{l',n'} c^{n'}_{l'}(1)\, \phi^{n'}_{1,l'} + \sum_{l',m'} d^{m'}_{l'} \psi^{m'}_{1,l'}, \tag{3.27}$$

where $m' = l''(N-1) + n'$, $l'' = 0, 1, ..., N-1$. Using (3.22), (3.25), and (3.26) we obtain after some algebra

$$c^n_l(1) = c^0_{n+Nl}(0), \tag{3.28}$$

and by (3.27)

$$d^{l'(N-1)+n'}_l(1) = c^{n'}_{l+l'}(0) - \sum_{l'',n''} c^0_{n''+Nl''}(0)\, p^{n''n'}_{l+l'-Nl''}, \tag{3.29}$$

where $n' = 1, ..., N-1$, $l' = 0, 1, ..., N-1$, l, $l'' \in \mathbb{Z}$. Note that (3.28) and (3.29) describe both the decomposition and reconstruction algorithm. How to proceed from level k to $k+1$, $k \in \mathbb{Z}$, should now be clear.

4. Examples

In this section we look at two concrete examples which will illustrate the above-introduced multiresolution analyses.

4.1. *Example* 1 $(V_k = \mathscr{V}_k \cap L^2(\mathbb{R}))$

Let u_i and v_i be given as in (2.5) with $s_i = s$, for all $i = 1, ..., N$. Note that (2.6) implies that

$$f^*(x) = sf^*(u_i^{-1}(x)) + c_i u_i^{-1}(x) + e_i, \; x \in [x_{i-1}, x_i], \tag{4.1}$$

$i = 1, ..., N$.

To calculate the inner-product of two FIF's we need the following result.

Proposition. *Let f^* and g^* be two* FIF's *interpolating* $\{(x_j, y_j): j = 0, 1, ..., N\}$ *and* $\{(x_j, \hat{y}_j): j = 0, 1, ..., N\}$, *respectively. Assume that* $x_0 = 0$ *and* $x_N = 1$. *Then*

$$\int_0^1 f^*(x)\, g^*(x)\, dx = \frac{\begin{bmatrix} \sum_{i=1}^n a_i[(s_i\hat{c}_i M_1 + s_i\hat{e}_i M_0 + \hat{s}_i c_i \hat{M}_1 + \hat{s}_i e_i \hat{M}_0) \\ + (c_i + \hat{c}_i)/3 + (c_i\hat{e}_i + e_i\hat{c}_i)/2 + e_i\hat{e}_i] \end{bmatrix}}{1 - \sum_i a_i s_i \hat{s}_i}, \tag{4.2}$$

where

$$M_0 = \int_0^1 f^*(x)\, dx = \frac{\sum_i a_i((c_i/2) + e_i)}{1 - \sum a_i s_i},$$

$$M_1 = \int_0^1 xf^*(x)\, dx = \frac{\sum_i a_i(d_i s_i M_0 + (a_i c_i)/3 + (d_i c_i + a_i e_i)/2 + d_i e_i)}{1 - \sum_i a_i^2 s_i},$$

and $\hat{M}_0$ and $\hat{M}_1$ are the corresponding moments of y^*.

Remark. $\hat{}$ refers to g.

Proof. The proof makes use of (4.1).

$$\int_0^1 f^*(x)\, g^*(x)\, dx$$
$$= \sum_i \int_{x_{i-1}}^{x_i} f^*(x)\, g^*(x)\, dx$$
$$= \sum_i \int_{x_{i-1}}^{x_i} v_i(u_i^{-1}(x), f^*(u_i^{-1}(x)))\, \hat{v}_i(u_i^{-1}(x), g^*(u_i^{-1}(x)))\, dx.$$

Now let $\xi_i = u_i^{-1}(x)$. Then the above integral reduces to

$$\sum_i \int_0^1 v_i(\xi_i, f^*(\xi_i))\, \hat{v}_i(\xi_i, g^*(\xi_i))\, a_i\, d\xi_i.$$

After some considerable algebra, we arrive at (4.2). ∎

Using the above proposition we now obtain an orthonormal basis.

Let $N=2$, $s=\frac{1}{2}$, and let $e^0 = (1, 0, 0)_{[0,1]}$, $e^1 = (0, 1, 0)_{[0,1]}$, and $e^2 = (0, 0, 1)_{[0,1]}$. Set $\tilde{\phi}^0 = e^1$, $\tilde{\phi}^1 = e^0 - e^2$, and determine $\tilde{\phi}^2$ by requiring

$$\langle \tilde{\phi}^2, \tilde{\phi}^0 \rangle = \langle \tilde{\phi}^2, \tilde{\phi}^1 \rangle = 0. \tag{4.3}$$

These conditions yield

$$\tilde{\phi}^2 = e^0 - \frac{2\langle e^0, e^1 \rangle}{\langle e^1, e^1 \rangle}\, e^1 + e^2. \tag{4.4}$$

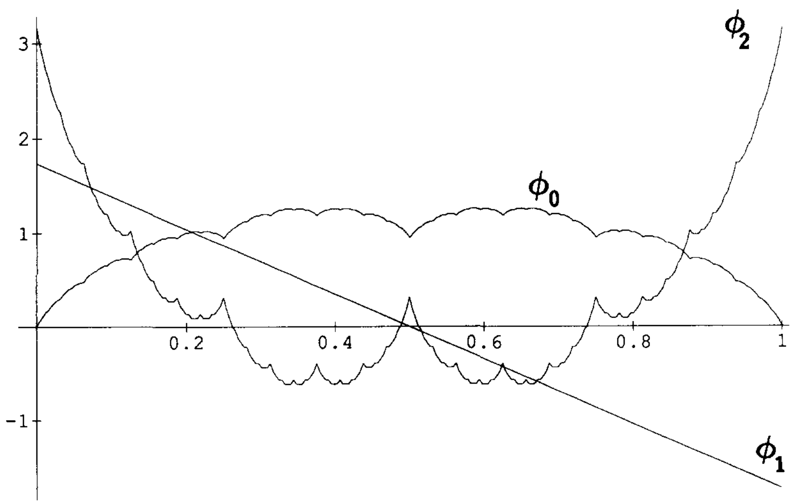

FIG. 2. The function $\{\phi^0, \phi^1, \phi^2\}$ in Example 1.

Normalizing $\{\tilde{\phi}^0, \tilde{\phi}^1, \tilde{\phi}^2\}$ yields the functions $\{\phi^0, \phi^1, \phi^2\}$ depicted below in Fig. 2.

Figure 3 shows 128 values obtained from a row of a digitized photograph plotted at $0, \frac{1}{2}, 1, \frac{3}{2}, ..., 64$.

Let f be the unique function in V_0 that interpolates the data on $[0, 64]$ and vanishes outside this interval, and let $c(0)$ be the coefficients in the expansion of f as in (3.7). Note that for $l<0$ or $l>64$ $c_l^n(0)=0$. Using the pyramid scheme (3.11) and (3.16) we decompose the data into

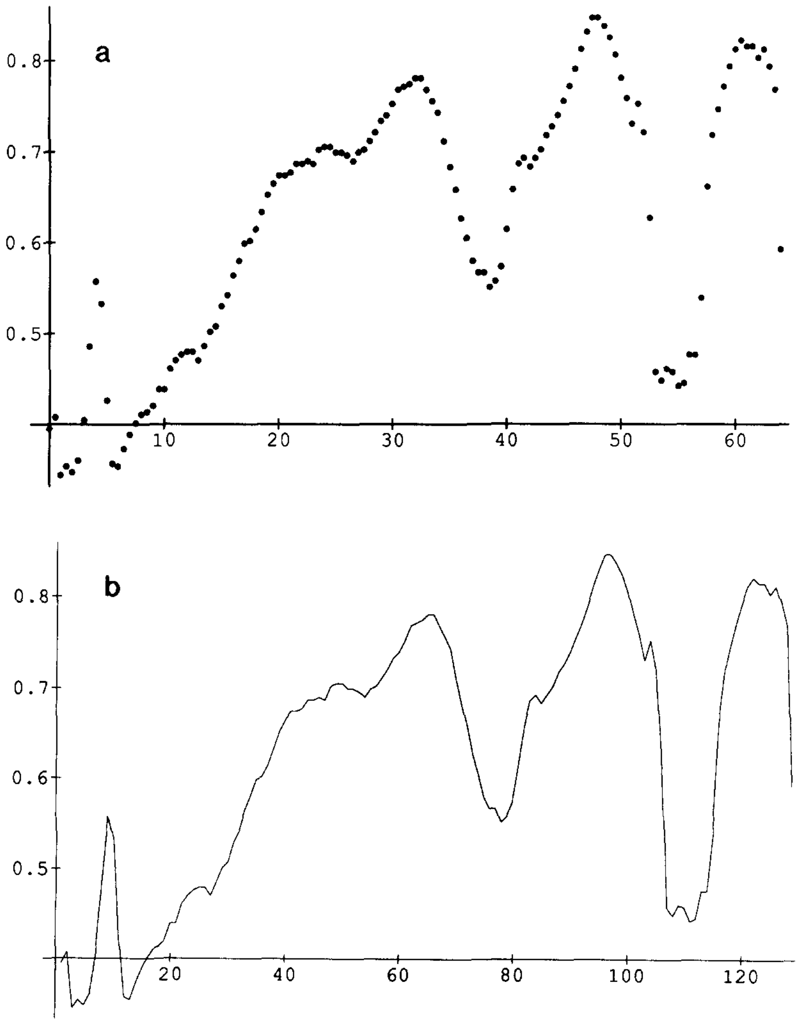

FIG. 3. (a) The data used in Examples 1 and 2; (b) the above data points linearly interpolated.

$d(1), \ldots, d(6)$, and $c(6)$. For illustrative purposes we filter the decomposed data by setting to zero the d-coefficients with $|d_l^n(k)| < 0.02$ and $|d_l^n(k)| < 0.01$, respectively. Applying (3.18) we reconstruct the filtered coefficients $\tilde{c}(0)$ from which we obtain the filtered data as shown in Figs. 4 and 5, respectively.

4.2. *Example* 2 $(V_k = \mathscr{V}_k \cap C_0(\mathbb{R}))$

Let us again choose $N = 2$ and $s = \frac{1}{2}$. We will reconstruct the same data as in Example 1 above.

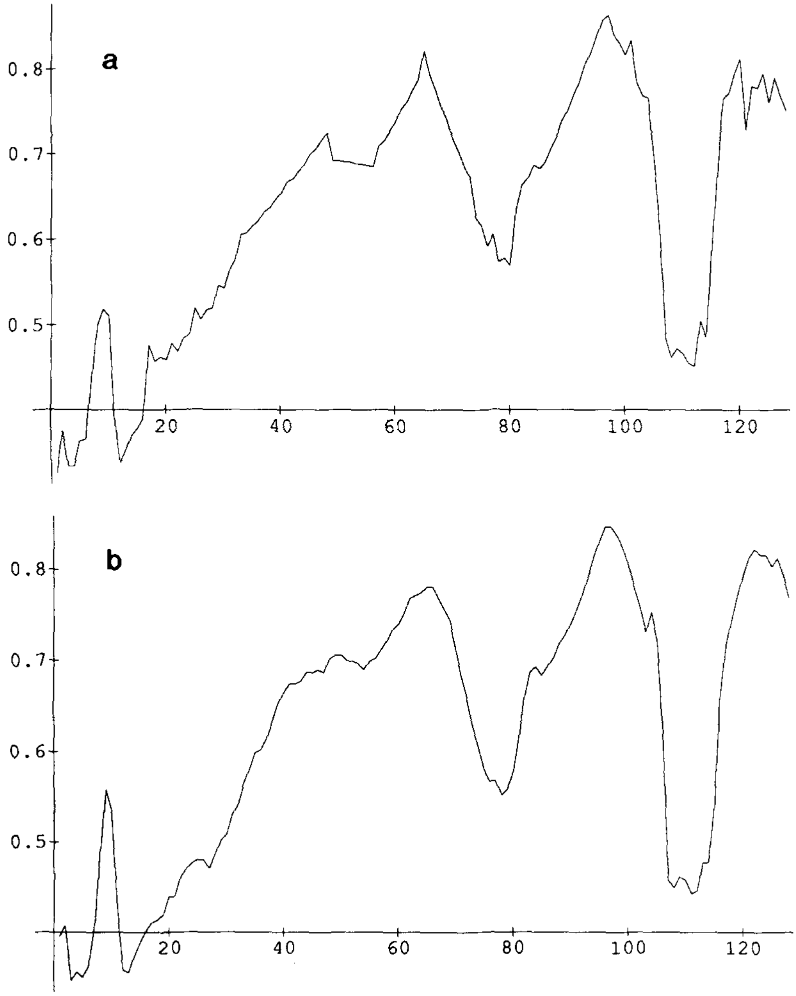

FIG. 4. (a) The reconstructed data with $|d(k)| < 0.02$ deleted; (b) the reconstructed data with $|d(k)| < 0.01$ deleted.

Let us briefly illustrate how this analysis works. Suppose $\{y_0, ..., y_4\}$ is the set of y-values for five equally spaced data points on $[0, 1]$. If $(y_0, y_1, y_2)_{[0, 1/2]}$ and $(y_2, y_3, y_4)_{[1/2, 1]}$ are the FIF's on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, respectively, then $(y_0, y_2, y_4)_{[0, 1]}$ is the FIF on $[0, 1]$. Furthermore, $(0, d_1, 0)_{[0, 1/2]}$ and $(0, d_2, 0)_{[1/2, 1]}$ is the wavelet function on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, respectively, where d_1 and d_2 are given by

$$\begin{aligned} d_1 &= y_1 - \frac{(1-s)\, y_0 + (1+2s)\, y_2 - s y_4}{2} \\ d_2 &= y_3 - \frac{(-s)\, y_0 + (1+2s)\, y_2 + (1-s)\, y_4}{2}. \end{aligned} \tag{4.5}$$

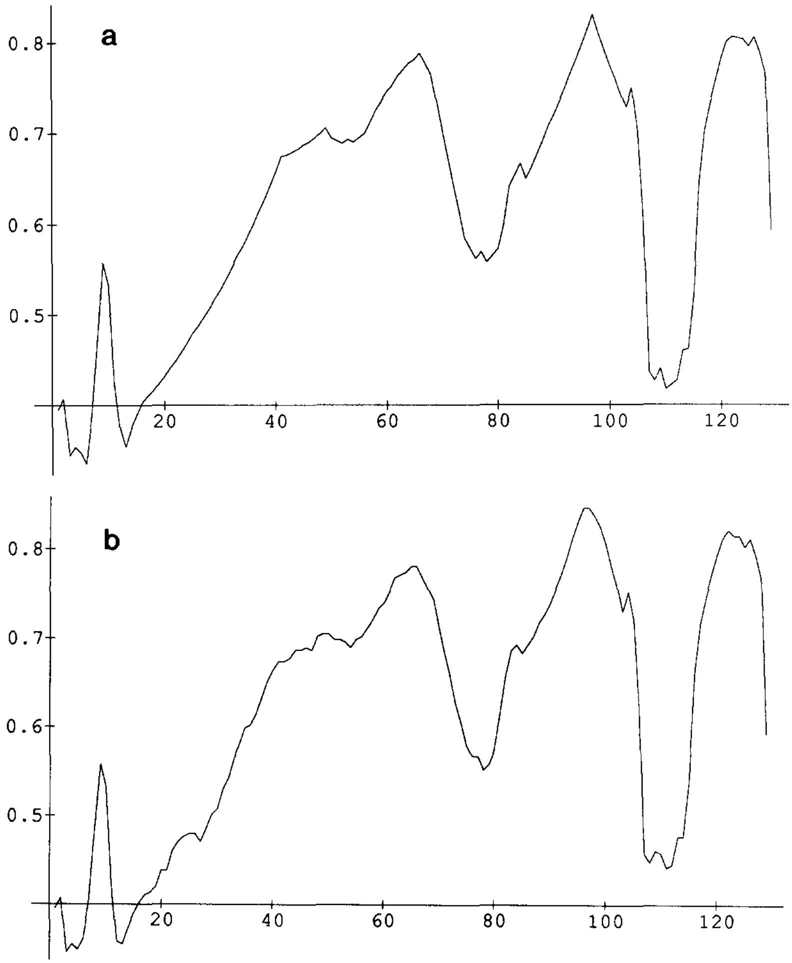

FIG. 5. (a) The reconstructed data with $|d(k)| < 0.02$ deleted; (b) the reconstructed data with $|d(k)| < 0.01$ deleted.

Figure 5 shows this reconstructed data $\tilde{c}(0)$ from $c(6)$ and $d(6)$, again deleting all the $d_l(k)$ with $|d_l(k)| < 0.02$ and $|d_l(k)| < 0.01$, respectively.

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References

1. M. F. Barnsley, Fractal functions and interpolation, *Constr. Approx.* **2** (1986), 303–329.
2. M. F. Barnsley, J. H. Elton, D. P. Hardin, and P. R. Massopust, Hidden-variable fractal interpolation functions, *SIAM J. Math. Anal.* **20** (1989), 1218–1242.
3. D. P. Hardin and P. R. Massopust, The capacity for a class of fractal functions, *Comm. Math. Phys.* **105** (1986), 455–460.
4. J. Hutchinson, Fractals and self-simularity, *Indiana Univ. Math. J.* **30** (1981), 713–747.
5. I. Daubechies, Orthonormal bases for compactly supported wavelets, *Comm. Pure Appl. Math.* **41** (1988), 909–996.
6. S. G. Mallat, Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$, *Trans. Amer. Math. Soc.* **315** (1989), 69–87.
7. Y. Meyer, Ondelettes et functions splines, *in* "Seminaire Equations aux Derivées Partielles, École Polytechnique, Paris, Dec. 1986."
8. P. Burt and E. Adelson, The Laplacian pyramid as a compact image code, *IEEE Trans. Comm.* **31** (1983), 482–540.